

# Traceless stress-energy and traversable wormholes

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## Abstract

A one-parameter family of static and spherically symmetric solutions to Einstein equations with a traceless energy-momentum tensor is found. When the nonzero parameter  $\beta$  lies in the open interval  $(0, 1)$  one obtains traversable Lorentzian wormholes. One also obtains naked singularities when either  $\beta < 0$  or  $\beta > 1$  and the Schwarzschild black hole for  $\beta = 1$ .

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In previous papers [1] we have rewritten the exterior and interior Schwarzschild solutions replacing the usual radial standard coordinate  $r$  with an angular one  $\psi$  defined by

$$r = \frac{2m}{\cos^2 \psi} \quad -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2} \quad (1)$$

when  $r > 2m$ , and analytically continued to

$$r = \frac{2m}{\cosh^2 \psi} \quad -\infty < \psi < \infty \quad (2)$$

when  $r < 2m$ ,  $m$  representing the gravitational mass of the considered body. This allows to obtain some results otherwise less apparent or even hidden in the usual coordinate systems. With special emphasis on the interior Schwarzschild solution, we introduced the concept of quasi-universe, namely an universe deprived of a spherical void. Two quasi-universes with the same gravitational mass are connected by the Einstein-Rosen bridge (fig.1) which can therefore be renamed an extreme wormhole., extreme because it is not seen as traversable by a static observer. The throat of the Einstein-Rosen bridge contains no matter and by fact

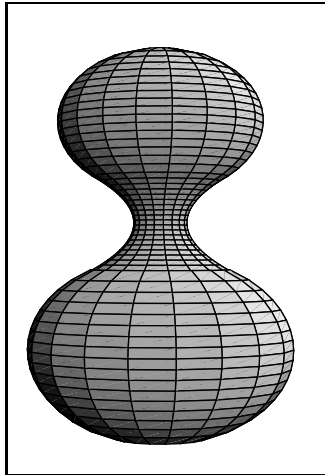


FIG. 1: The connection between two quasi-universes.

constitutes the  $I \cup III$  representation of the exterior Schwarzschild solution in the customary nomenclature of the Kruskal-Szekeres diagram [2]. We interpreted this representation as describing two sources of equal gravitational mass placed at the boundaries  $\psi = \pi/2$  and  $\psi = -\pi/2$ , so as a limiting case of fig.1 when the connections between the two quasi-universes and the Einstein-Rosen bridge tend to infinity. The quasi-universes can exchange information if the Einstein-Rosen bridge is substituted by a two-way traversable wormhole. After the seminal paper of Morris and Thorne [3] there was a considerable amount of activity

about wormhole physics [4,5,6,7,8,9,10]. The prescriptions to have a wormhole traversable are clearly given in [3,4], and particularly important with this respect is the violation of the null energy condition (NEC) [11]. In our opinion to have a wormhole connecting two quasi-universes it is also necessary that no singularity lies beyond its throat. This fact can occur, as an example, in the Brans-Dicke theory of gravitation when the post-Newtonian parameter  $\gamma$  is greater than unity. In this case one effectively obtains a solution describing a wormhole which connects two spaces asymptotically flat [7,12], but obtains also another solution where beyond the throat there is a singularity smeared on a spherical surface asymptotically large but not asymptotically flat [12,13,14]. The aim of this paper is to find a wormhole solution without singularities beyond the throat and different from the Einstein-Rosen bridge only by the occurrence of a “vacuum tension”, which implies the presence in the throat of exotic matter violating the NEC condition. Having in mind the work of Kar and Sahdev [6], where a class of wormhole solutions is achieved starting from a choice of the “potential”  $g_{tt}$  and obtaining  $g_{rr}$  as a solution to the constraint of vanishing Ricci scalar curvature ( $R = 0$ ), we find the metric components by assuming a traceless energy-momentum tensor ( $T = 0$ ). Our ansatz implies again  $R = 0$  but the constraints are now on the energy density  $\rho$  and on the radial and transverse pressures  $p_{\parallel}$  and  $p_{\perp}$ :

$$\begin{cases} \rho = 0 \\ p_{\parallel} + 2p_{\perp} = 0 \end{cases} \quad (3)$$

(Note that sometimes in the literature the radial pressure is substituted by minus the radial tension  $\tau$  and the transverse pressure is simply denoted by  $p$ ).

Let us firstly consider the static spherically symmetric line element

$$ds^2 = A(r)dr^2 + r^2 d\Omega^2 - B(r)dt^2 \quad (4)$$

where

$$d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \quad (5)$$

and the energy-momentum tensor with components

$$T_r^r = p_{\parallel}(r), \quad T_{\vartheta}^{\vartheta} = T_{\varphi}^{\varphi} = p_{\perp}(r), \quad T_t^t = -\rho(r) \quad (6)$$

The relevant Einstein equations, dropping for simplicity the  $r$  dependence and denoting by a prime the derivative with respect to  $r$ , are in units  $c = G = 1$

$$\frac{B - AB + rB'}{r^2 AB} = 8\pi p_{\parallel} \quad (7a)$$

$$- \frac{1}{4rA^2B^2} \{ 2B^2A' + rAB'^2 + B [ rA'B' - 2A ( B' + rB'' ) ] \} = 8\pi p_{\perp} \quad (7b)$$

$$\frac{A - A^2 - rA'}{r^2 A^2} = -8\pi \rho \quad (7c)$$

Taking into account the constraints (3) one obtains from eqs. (7)

$$A = \frac{1}{1 - \frac{2\eta}{r}} \quad (8)$$

$$B = \left[ \frac{1 + \beta \left( \sqrt{1 - \frac{2\eta}{r}} - 1 \right)}{\alpha} \right]^2 \quad (9)$$

where  $\eta$ ,  $\alpha$  and  $\beta$  are constants. The constant  $\alpha$  can be related to the position  $r_{obs}$  of the observer [1], while  $\eta$  and  $\beta$  are related to the gravitational mass  $m$  by comparison with the Newtonian limit at large distances. More in detail, if we put

$$\alpha = 1 + \beta \left( \sqrt{1 - \frac{2\eta}{r_{obs}}} - 1 \right), \quad \eta = \frac{m}{\beta} \quad (10)$$

the previous expressions for  $A$  and  $B$  become

$$A = \frac{1}{1 - \frac{2m}{\beta r}} \quad (11)$$

$$B = \left[ \frac{1 + \beta \left( \sqrt{1 - \frac{2m}{\beta r}} - 1 \right)}{1 + \beta \left( \sqrt{1 - \frac{2m}{\beta r_{obs}}} - 1 \right)} \right]^2 \quad (12)$$

so, when the observer is put at spatial infinity, the asymptotic behaviour of  $B$  is effectively  $B \approx 1 - 2m/r$  regardless of the sign of  $\beta$ . As to the pressures, they are

$$p_{\parallel} = -2p_{\perp} = \frac{(-1 + \beta)m}{4\pi\beta \left[ 1 + \beta \left( \sqrt{1 - \frac{2m}{\beta r}} - 1 \right) \right] r^3} \quad (13)$$

and are independent, as they must be, on the observer position  $r_{obs}$ . Therefore once  $r_{obs}$  is fixed in eq. (12), the nonzero constant  $\beta$  becomes the only free parameter in this model. A treatment of the integration constants different from ours is given in [15].

The invariant of curvature  $K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ , which will be needed in the following, is given by

$$K = \frac{24m^2}{r^7\beta^2 \left[ 1 + \beta \left( \sqrt{1 - \frac{2m}{\beta r}} - 1 \right) \right]^2} \left\{ -4m\beta + \right. \\ \left. r \left[ 1 + \beta \left( 3\beta - 2(-1 + \beta) \sqrt{1 - \frac{2m}{\beta r}} - 2 \right) \right] \right\} \quad (14)$$

The lapse function  $B$  vanishes at  $r_* = 2m\beta/(2\beta - 1)$  when  $\beta < 0$  or when  $\beta \geq 1$ . The particular value  $\beta = 1$  describes a black hole in Schwarzschild geometry with zero pressures, event horizon at  $r_h = 2m$ , curvature invariant given by  $48m^2/r^6$  and therefore essential singularity at  $r = 0$ . If instead  $\beta < 0$  or  $\beta > 1$  both the pressures and the curvature invariant  $K$  diverge at  $r = r_*$  where  $B$  vanishes so we have no more an event horizon but a naked singularity at  $r_*$ ; let us notice that the essential singularity occurs now at a value of  $r$  greater than the value  $r_0 = 2m/\beta$  where otherwise should blow up the function  $A$  if  $\beta$  is nonnegative.

Let us consider the case  $0 < \beta < 1$ . Now the function  $B$  does not vanish and is everywhere finite so if we introduce the proper length  $l$  through

$$l = \pm \int_{r_0}^r \frac{dr'}{\sqrt{1 - \frac{2m}{\beta r'}}} = \pm \left\{ r \sqrt{1 - \frac{r_0}{r}} + r_0 \log \left[ \sqrt{\frac{r}{r_0}} \left( 1 + \sqrt{1 - \frac{r_0}{r}} \right) \right] \right\} \quad (15)$$

and put the spacetime metric (4) in the form

$$ds^2 = dl^2 + r^2(l)d\Omega^2 - e^{2\phi(l)} dt^2 \quad (16)$$

where  $\phi(l) = \log B[r(l)]$ , it is apparent that the above line element satisfies all the properties required to can describe a Lorentzian wormhole with radius of the throat given by  $r_0 = 2m/\beta$  and traversable in principle because of the absence of an event horizon. Also the invariant of curvature  $K$  and the pressures are everywhere finite, in particular the pressures at the throat are given by

$$p_{\parallel} = -2p_{\perp} = -\frac{m}{4\pi\beta r_0^3} = -\frac{\beta^2}{32\pi m^2} \quad (17)$$

Another coordinate patch which covers the entire geometry, besides the one given in eq. (16), and which allows to obtain somewhat simpler expressions can be obtained if we replace the standard radial coordinate  $r$  not by the proper length  $l$  but by an angular variable  $\psi$ , as discussed at the very beginning of the paper, which can be given in this case by

$$r = \frac{2m}{\beta \cos^2 \psi} \quad -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2} \quad (18)$$

The spacetime metric (4) then becomes

$$ds^2 = \left( \frac{4m}{\beta \cos^3 \psi} \right)^2 d\psi^2 + \left( \frac{2m}{\beta \cos^2 \psi} \right)^2 d\Omega^2 - \left[ \frac{1 + \beta (|\sin \psi| - 1)}{1 + \beta (|\sin \psi_{obs}| - 1)} \right]^2 dt^2 \quad (19)$$

and, because of traversability, the observer could even be put at the throat where, according to (18),  $\psi_{obs} = 0$ . Therefore two quasi-universes can be connected by a traversable wormhole though at the expense of violation of the energy conditions.

Leaving apart discussions about the traversability in practice or about known violations of the energy conditions, we would like to make some final remarks. We remind that our one-parameter solution depends so critically on the parameter  $\beta$  as to pass at  $\beta = 1$  from a naked singularity ( $\beta > 1$ ) to a wormhole ( $0 < \beta < 1$ ) through a state describing a Schwarzschild black hole ( $\beta = 1$ ). We found an analogous behavior [7] when considering the static spherically symmetric vacuum solution of the Brans-Dicke theory of gravitation: in that case by varying the post Newtonian parameter  $\gamma$  different kinds of solutions were obtained when passing through the critical value  $\gamma = 1$ . This value corresponds to the usual Schwarzschild exterior solution and a possible physical interpretation was given [12] by introducing spacetime fluctuations at a scale comparable to Planck length near the event horizon of a black hole. Another explanation, which seems worthy to be further investigated, might reside on the high nonlinearity of Einstein equations which makes solutions to change qualitatively, as in the theory of dynamical systems, when a reference state loses its stability because a “control parameter”, in the actual case  $\beta$ , passes through some critical value.

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